REPORT No. 200

SOME PROBLEMS ON THE LIFT AND ROLLING MOMENT OF AIRPLANE WINGS

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1. INTRODUCTION

This paper is part of the thesis submitted by the writer to Johns Hopkins University for the degree of doctor of philosophy. The Committee on Aerodynamics has approved its publication as a technical report of the National Advisory Committee for Aeronautics.

Aerodynamics is largely an experimental science, but certain of its problems have been brought within the scope of mathematical analysis. This is especially true with regard to wing profiles, airship bodies, and lifting surfaces. The most fruitful method for the mathematical investigation of lifting surfaces is the airfoil theory originated by Prandtl[1] and Munk[2] and extended in various directions by them and others, particularly Lieutenant Betz.[3] The present paper is concerned for the most part with the application of the airfoil and twisted wing theory to the calculation of the lift and rolling moment of airplane wings. Most of the results arrived at are strictly true only for wings of elliptic plan form. The following investigation aims to give some indications of the accuracy with which the results can be applied to the wing forms in actual use.

On account of the importance of Munk's twisted wing theory, it has been thought worth while to give an outline of it before applying the method to the problems treated.

ASSUMPTIONS, DEFINITIONS, AND FUNDAMENTAL FORMULAS

In this paper the air will be considered as a frictionless, incompressible fluid of constant density.

The velocity of flight relative to the undisturbed air will be assumed constant in magnitude and direction. It will be denoted by V, and for convenience supposed to be horizontal.

Since it is frequently necessary to replace sines and tangents by their arcs in the applications of the airfoil theory, it is useless to attempt more than three-figure accuracy in numerical results. All numerical results will therefore be given to only three significant figures.

The *aspect ratio* of a wing is defined as the ratio of the span to the average chord. Denoting aspect ratio by a, we have

$$a=\frac{2l}{\text{average chord}}=\frac{2l}{\frac{\text{area}}{2l}}=\frac{4l^2}{\text{area}}$$

Hence,

$$a=\frac{8l}{\pi c_0}\text{ for an elliptic wing}$$

$$=\frac{2l}{c_0}\text{ for a rectangular wing,}$$

where $2l$ = length of wing, and

c_0 = maximum chord of the wing.

[1] L. Prandtl: "Tragflügeltheorie" I and II Mitteilung. Nachrichten von der Kgl. Gesellschaft der Wissenschaften. Math. Phys. Klasse Göttingen, 1918, 1919.

[2] M. Munk: "Beitrag zur Aerodynamik der Flugzeugtragorgane." Technische Berichte der Flugzeugmeisterei, Bd. II, 1918. "Isoperimetrische Aufgaben aus der Theorie des Fluges." Dissertation, Göttingen, 1919. Translated into English as "The Minimum Induced Drag of Airfoils," Washington, 1921. N. A. C. A. Technical Report No. 121. "The Twisted Wing with Elliptic Plan Form." Technical Note No. 109. National Advisory Committee for Aeronautics, Washington, 1922. "General Theory of Thin Wing Sections," National Advisory Committee for Aeronautics Technical Report No. 142, 1922

[3] A. Betz: "Beiträge zur Tragflügeltheorie mit besonderer Berücksichtigung des einfachen Rechteckigen Flügels" Dissertation, Göttingen, 1919.

In order to simplify the analytical treatment, the aspect ratio is always assumed to be large, so that the wing is replaced by a "lifting line." The "downwash speed" at a point x of the lifting line is the mean velocity component of the air flow in the vicinity of the point x, at right angles to the velocity of flight, taken, say, along a small circle around the lifting line as axis. Its magnitude can be computed from the distribution of the vortices running off from the trailing (rear) edge of the wing and is given by the formula [4]

$$w=\frac{1}{4\pi}\int_{-l}^{+l}\frac{d\Gamma}{dx}\frac{dx}{x'-x} \qquad (1)$$

where w denotes the magnitude of the downwash speed and Γ is the circulation around the wing at the point $x=x'$.

The lift is the component of the air force at right angles to the velocity of flight. The entire lift of the wing is denoted by L.

The *density of lift*, or intensity of the lift, is the lift per unit length of wing. It will be denoted by L', and its value is $\frac{dL}{dx}$. The relation between density of lift and circulation around the wing segment at all points is given by the formula [5]

$$L'=\frac{dL}{dx}=\rho V\Gamma,$$

where ρ is the density of the air and V is the velocity of flight. In general the circulation is variable along the span of the wing and drops to zero at its ends. Hence Γ is a function of x and furthermore one which we may differentiate with respect to x. Taking the derivative of L' with respect to x, we get

$$\frac{dL'}{dx}=\rho V\frac{d\Gamma}{dx}, \text{ or } \frac{d\Gamma}{dx}=\frac{1}{\rho V}\frac{dL'}{dx}$$

Substituting this value of $\frac{d\Gamma}{dx}$ in the formula (1) for downwash speed w, we get

$$w=\frac{1}{4\pi\rho V}\int_{-l}^{+l}\frac{dL'}{dx}\frac{dx}{x'-x} \qquad (2)$$

In an infinitely long wing (with all cross sections equal and parallel) the density of lift is constant along the wing and the circulation is the same at all points. Hence $\frac{dL'}{dx}=0$, and $w=0$ by Equation (2). We thus see that the downwash speed is zero for an infinite, cylindrical wing.

The *geometric angle of attack* at any point along the wing span is the acute angle which the chord of the wing at that point makes with the direction of flight relative to the undisturbed air. It will be denoted by α_g and is usually variable along the span.

The *induced angle of attack* at any point is the mean angle through which the air at that point is deflected downward. Its value is arc tan $\frac{w}{V}$, and it will be denoted by α_i. We usually write

$$\alpha_i=\frac{w}{V} \qquad (3)$$

The *effective angle of attack* α_e at any point is the acute angle which a line fixed with respect to the section at that point makes with the resultant current of undisturbed air. The direction of this line is so chosen as to give zero lift at the angle of attack zero. In the case of an infinitely long cylindrical wing, it is the same as the geometric angle of attack. For a wing of finite length its value is

$$\alpha_e=\alpha_g-\alpha_i \qquad (4)$$

[4] Prandtl, Applications of Modern Hydrodynamics to Aeronautics, N. A. C. A. Technical Report No. 116, p. 37, 1921

[5] Prandtl, Applications of Modern Hydrodynamics to Aeronautics, p. 49.

The *drag* acting on a wing segment is the component of the air force acting on the segment parallel to the direction of motion. It is generally a function of x and the whole drag of the entire wing will be denoted by D. The drag which is considered in the airfoil theory can be computed by first computing the downwash speed w.

Since the air force on a wing segment is always directed at right angles to the relative motion between the segment and the air, it is not directed at right angles to the direction of motion but inclined toward it by the angle $\frac{w}{V}$, thus having a horizontal component.[6] Hence the relation between lift and drag is given by the formula

$$D=\frac{w}{V}L \qquad (5)$$

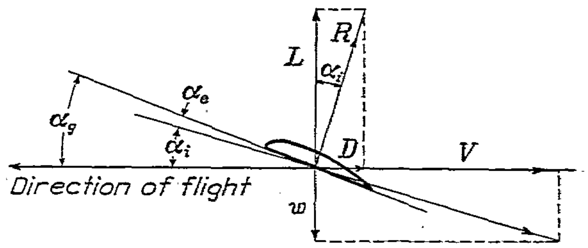

FIG. 1

The relations between the lift and drag and the three angles of attack are illustrated by Figure 1.

The relation between density of lift L' and wing form c is given by the formula

$$L'=2\pi\ \alpha_e\ qc \qquad (6)$$

where c is the length of the chord at the point considered, α_e is the effective angle of attack at that point counted from the angle of zero lift, and q is the dynamic pressure$=\frac{1}{2}\rho V^2$.

We can derive formula (6) as follows:

$$L'=\rho V\Gamma$$

Prandtl writes [7]

$$\Gamma=Vc(c_1\alpha_e+c_2)$$

and states [7] that $c_1=\pi$. Substituting this in the formula $L'=\rho V\Gamma$, we get

$$L'=\rho V^2c(\pi\alpha_e+c_2)$$

Defining α_e so that $L'=0$ when $\alpha_e=0$, we get $c_2=0$ and then

$$L'=\rho V^2c\pi\alpha_e=2\pi\ \alpha_e\cdot\frac{1}{2}\rho V^2c=2\pi\ \alpha_e\ qc.$$

The contribution to the *rolling moment* at any point along the wing is the moment of the intensity of lift (at that point) about the longitudinal axis of the wing. Denoting it by M, we have for the whole wing

$$M=\int_{-l}^{+l}xL'dx=2\pi q\int_{-l}^{+l}\alpha_e\ cx\ dx.$$

[6] Prandtl, Applications of Modern Hydrodynamics to Aeronautics, N. A. C. A. Report No. 116, 1921, p. 36.
[7] Prandtl, N. A. C. A. Report No. 116, p. 41, 1921.

2. THEORY OF THE TWISTED ELLIPTIC WING

By a twisted wing is meant a warped wing or one in which the geometric angle of attack varies from point to point along the span.

Since all vortices created by the motion of the lifting line are concentrated in a surface approximately plane and since these vortices are approximately straight and parallel to the motion, the components of the air flow in a vertical plane through the lifting line can be assumed to be a two-dimensional potential flow, with all singularities at the points of the lifting line. The velocity of downwash is therefore derivable from a velocity potential, and hence the method of the theory of functions of a complex variable are applicable. These facts, together with an important theorem (stated later on p. 336) connecting lift and velocity-potential, led Munk[8] to write the complex function

$$\Sigma F = F_n \qquad (7)$$

$$F_n = \varphi + i\psi = iB_n Vl \left(\frac{z}{l} + i\sqrt{1-\left(\frac{z}{l}\right)^2}\right)^n \qquad (8)$$

where n denotes any positive integer, B_n a dimensionless constant, V the velocity of flight, and z the complex number $x+iy$. The x-axis is taken along the wing length, the y-axis is vertical, and the length of the wing is taken as $2l$. The wing tips are therefore the points $z = \pm l$.

It is to be observed that at all points except $z = \pm l$, F_n is a double-valued function of z, each value being a single-valued analytic function of z.

The substitution

$$z = l \cos \theta \qquad (9)$$

makes equation (8) assume the more convenient form

$$F_n = iB_n Vl\ (\cos \theta \pm i \sin \theta)^n \qquad (10)$$

The two values of F are therefore

$$F' = \varphi_1 + i\psi_1 = iBVl\ (\cos \theta + i \sin \theta)^n = iBVl\ (\cos n\theta + i \sin n\theta) \qquad (10a)$$

$$F'' = \varphi_2 + i\psi_2 = iBVl\ (+\cos \theta - i \sin \theta)^n = iBVl\ (\cos n\theta - i \sin n\theta) \qquad (10b)$$

θ is real at points along the axis of reals between $-l$ and l, that is, *along the wing*, and these are the only points in which we are interested. Hence the two values of F along the wing are

$$F_1 = \varphi_1 + i\psi_1 = iBVl\ (\cos n\theta + i \sin n\theta)$$

and

$$F_2 = \varphi_2 + i\psi_2 = iBVl\ (\cos n\theta - i \sin n\theta)$$

These equations show that F is discontinuous along the wing and that the amount of the discontinuity is

$$\varphi_2 - \varphi_1 = 2\ BVl \sin n\theta \qquad (11)$$

The *density of lift* can now be found by means of the following theorem due to Dr. Munk.[9]

The density of lift perpendicular to the lifting line is proportional to the discontinuity of the velocity-potential and has the value

$$L' = 2\rho V\ (\varphi_2 - \varphi_1)$$

where φ denotes velocity-potential and $\varphi_2 - \varphi_1$ is the difference of velocity-potential on opposite sides of the wing. Hence the density of lift is here

$$L' = 4\rho BV^2 l \sin n\ \theta = 8Bql \sin n\theta \qquad (12)$$

where

$$q = \frac{1}{2}\rho\ V^2$$

[8] Max M. Munk, "Elements of the Wing Section Theory and Wing Theory." Technical Report No. 191, N. A. C. A., 1924.

[9] "The Minimum Induced Drag of Aerofoils." Technical Report No. 121, p. 14, N. A. C. A., 1921.

For the points along the wing we have

$$x=l\cos\theta,\text{ from (9).}$$

Substituting this value of x in the equation of the wing plan form,

$$\frac{x^2}{l^2}+\frac{c^2}{c_0^2}=1,$$

we get $c=c_0\sin\theta$. Recalling now the formula

$$L'=2\pi\alpha_e qc,$$

we have

$$2\pi\,\alpha_e\,c_0\sin\theta=8\,Bl\sin n\theta$$

or

$$\alpha_e=\frac{4Bl\sin n\theta}{\pi c_0\sin\theta}\text{ --------------------------------(13)}$$

To find the downwash speed w and therefore the induced angle of attack α_i we must calculate the partial derivative $\frac{\partial\varphi}{\partial y}$. This is most easily done by differentiating equation (10) with respect to z by means of the formula

$$\frac{dF}{dz}=\frac{dF}{d\theta}\cdot\frac{d\theta}{dz}$$

Reference to Equation (11) and Munk's theorem for density of lift shows that in order to obtain lift we must take φ for velocity-potential. Then we have from (10), remembering that

$$\frac{dF}{dz}=\frac{\partial\varphi}{\partial x}-i\frac{\partial\varphi}{\partial y},$$

$$-\frac{\partial\varphi}{\partial y}=+w=+\frac{BV\,n\sin n\theta}{\sin\theta}$$

Hence the induced angle of attack is

$$\alpha_i=\frac{w}{V}=\frac{BV\,n\sin n\theta}{\sin\theta}\text{ ----------------------------(14)}$$

and therefore

$$\frac{\alpha_e}{\alpha_i}=\frac{4\,l}{\pi c_0 n}\text{ --------------------------------(14a)}$$

for all points along the wing.

Hence, for a given value of n the ratio of the effective to the induced angle of attack is the same for all points along the wing span.

It is the simple character of the distributions of lift having this property which simplifies the present problem for the special case of the elliptic plan view.

Since the geometric angle of attack is the sum of the effective and induced angles of attack, we get

$$\alpha_g=\alpha_e+\alpha_i=\alpha_e+\frac{\pi c_0 n}{4l}\,\alpha_e=\left(1+\frac{\pi c_0 n}{4l}\right)\alpha_e$$

Substituting the value of α_e from (13), we get

$$\alpha_g=\left(1+\frac{\pi c_0 n}{4l}\right)\frac{4Bl\sin n\theta}{\pi c_0\sin\theta}\text{ ---------------------------(15)}$$

Since

$$L'=2\pi\alpha_e qc$$

and

$$\alpha_e=\frac{\alpha_g}{1+\frac{\pi c_0 n}{4l}}$$

we have

$$L'=8\,Bql\sin n\theta$$

Thus far we have been dealing with an elliptic wing of "harmonic" twist, each value of n giving a different type of twist and large values of n giving more changes of direction of the twist than small values. Since the angle of attack and density of lift can always be expressed as a Fourier series, we can get the most general distribution of lift by giving n all positive values and superposing all the particular distributions. Hence for the geometric angle of attack we may write

$$\alpha_g = a_1\frac{\sin\theta}{\sin\theta} + a_2\frac{\sin 2\theta}{\sin\theta} + a_3\frac{\sin 3\theta}{\sin\theta} + \cdots + a_n\frac{\sin n\theta}{\sin\theta}$$
$$= (\alpha_g)_1 + (\alpha_g)_2 + \cdots + (\alpha_g)_n + \cdots$$

or

$$\alpha_g \sin\theta = f(\theta) = a_1 \sin\theta + a_2 \sin 2\theta + \cdots + a_n \sin n\theta + \cdots \tag{16}$$

The series on the right is a Fourier series, and if we assume α_g to be known we can determine the coefficients a_n by the usual method.

From the relation

$$\alpha_e = \frac{\alpha_g}{1+\frac{\pi c_0 n}{4l}}$$

we see that the effective angle of attack is

$$\alpha_e = \frac{a_1\frac{\sin\theta}{\sin\theta}}{1+K} + \frac{a_2\frac{\sin 2\theta}{\sin\theta}}{1+2K} + \cdots + \frac{a_n\frac{\sin n\theta}{\sin\theta}}{1+nK} + \cdots \tag{17}$$

where

$$K = \frac{\pi c_0}{4l}$$

The density of lift is therefore

$$L' = 2\pi\alpha_e\, q\, c$$
$$= 2\pi q c_0 \sin\theta\left(\frac{a_1\frac{\sin\theta}{\sin\theta}}{1+K} + \frac{a_2\frac{\sin 2\theta}{\sin\theta}}{1+2K} + \cdots\right)$$

or

$$L' = 2\pi q c_0\left(\frac{a_1 \sin\theta}{1+K} + \frac{a_2 \sin 2\theta}{1+2K} + \cdots + \frac{a_n \sin n\theta}{1+n\,K} + \cdots\right) \tag{18}$$

3. THE EFFECT OF DOWNWASH ON LIFT AND ROLLING MOMENT FOR ELLIPTIC AND RECTANGULAR WINGS

Downwash occurs with all wings of finite length. Its effect is to reduce the lift of the wing in a certain ratio. The magnitude of this ratio depends upon the form of the wing, the aspect ratio, and the variation of the geometric angle of attack along the wing span. To calculate the reducing effect of downwash on the lift and rolling moment in any given case, we calculate the true or actual lift and rolling moment from the effective angle of attack, thus taking account of downwash. Then we calculate a fictitious lift and rolling moment from the geometric angle of attack alone, neglecting the effect of the downwash. This fictitious or ideal lift is the lift that would be produced by a segment of an infinitely long cylindrical wing having the same geometric angle of attack, the length of the segment being equal to that of the actual wing. A similar statement applies to the fictitious rolling moment. The ratio of the true to the fictitious lift is a number less than unity, and we shall call this number the *downwash factor* for lift. Likewise, the ratio of the true to the fictitious rolling moment gives the downwash factor for rolling moment. We now proceed to calculate these factors for elliptic and rectangular wings.

ELLIPTIC WING

The density of lift for a twisted elliptic wing has already been found to be

$$L'=\frac{dL}{dx}=2\pi qc_0\left(\frac{a_1 \sin \theta}{1+K}+\frac{a_2 \sin 2\theta}{1+2K}+\cdot\cdot\cdot\cdot+\frac{a_n \sin n\theta}{1+nK}+\right).$$

Since $x=l \cos \theta$, $dx=-l \sin \theta \, d\theta$, we have

$$dL=-2\pi\, qc_0 l\left(\frac{a_1 \sin^2 \theta}{1+K}+\frac{a_2 \sin \theta \sin 2\theta}{1+2K}+\cdot\cdot\right)d\theta$$

Hence

$$L=2\pi qc_0 l\left(\frac{a_1}{1+K}\int_0^\pi \sin^2 \theta \, d\theta+\frac{a_2}{1+2K}\int_0^\pi \sin \theta \sin 2\theta \, d\theta+\cdot\cdot\right)$$

or

$$L=\frac{\pi^2 a_1 q c_0 l}{1+K} \quad\text{------------------------------}\quad (19)$$

The rolling moment is

$$M=\int x\, dL=2\pi\, q\, c_0\, l^2\int_0^\pi\left(\frac{a_1 \sin \theta}{1+K}+\frac{a_2 \sin 2\, \theta}{1+2K}+\right)\sin \theta \cos \theta \, d\theta$$

$$=\pi qc_0\, l^2\int_0^\pi\left(\frac{a_1 \sin \theta \sin 2\, \theta}{1+K}+\frac{a_2 \sin^2 2\, \theta}{1+2K}+\cdot\cdot\right)d\theta$$

or

$$M=\frac{\pi^2\, qc_0 l^2\, a_2}{2\,(1+2K)} \quad\text{------------------------------}\quad (20)$$

The geometric angle of attack for the twisted elliptic wing has likewise been found to be

$$\alpha_g=\frac{a_1 \sin \theta}{\sin \theta}+\frac{a_2 \sin 2\theta}{\sin \theta}+\cdot\cdot\cdot\cdot+\frac{a_n \sin n\theta}{\sin \theta}+$$

Treating this as the effective angle of attack and substituting it for α_e in the formula

$$L'=2\pi\, \alpha_e\, qc,$$

we get, since $c=c_0 \sin \theta$,

$$L'_g=\frac{dL_g}{dx}=2\pi\, qc_0\, \sin\, \theta\left(\frac{a_1 \sin \theta}{\sin \theta}+\frac{a_2 \sin 2\, \theta}{\sin \theta}+\cdot\cdot\right)$$

or

$$dL_g=-2\pi qc_0\, l\, (a_1 \sin^2 \theta+a_2 \sin \theta \sin 2\, \theta+\cdot\cdot\cdot\cdot\cdot)\, d\theta$$

Hence

$$L_g{}^{10}=2\pi qc_0\, l\int_0^\pi (a_1 \sin^2 \theta+a_2 \sin \theta \sin 2\theta+\cdot\cdot\cdot)\, d\theta$$

or

$$L_g=\pi^2\, qc_0\, l\, a_1 \quad\text{------------------------------}\quad (21)$$

The fictitious rolling moment is

$$M_g=\int x\, dL_g=\pi qc_0\, l^2\, \int_0^\pi (a_1 \sin \theta \sin 2\theta+a_2 \sin^2 2\theta+\cdot\cdot)\, d\theta$$

or

$$M_g=\frac{\pi^2\, qc_0 l^2\, a_2}{2} \quad\text{------------------------------}\quad (22)^{10}$$

[10] In the following pages we shall denote by L_g and M_g the fictitious lift and rolling moment due to the geometric angle of attack alone.

Dividing (19) by (21),

$$\frac{L}{L_g}=\frac{1}{1+K}$$

or

$$L=\frac{1}{1+K}L_g \quad \text{(23)}$$

From this last equation we get the theorem:

The true lift of a twisted elliptic wing can be found by neglecting the downwash, calculating the ideal or fictitious lift, and then multiplying it by the factor $\frac{1}{1+K}$.

This downwash factor is independent of the twist of the wing and depends only on the aspect ratio, as we shall now show. The number K stands for $\frac{\pi c_0}{4l}$. The aspect ratio for an elliptic wing has already been found to be $\frac{8l}{\pi c_0}$. Hence

$$a=\frac{8l}{\pi c_0}=2\left(\frac{4l}{\pi c_0}\right)=\frac{2}{K} \text{ or } K=\frac{2}{a}.$$

The downwash factor for lift in the case of an elliptic wing is therefore

$$f_1=\frac{1}{1+K}=\frac{1}{1+\frac{2}{a}} \quad \text{(24)}$$

Dividing (20) by (22), we get

$$f_2=\frac{M}{M_g}=\frac{1}{1+2K}=\frac{1}{1+\frac{4}{a}} \quad \text{(25)}$$

or

$$M=\left(\frac{1}{1+\frac{4}{a}}\right)M_g$$

We thus get a method for calculating the rolling moment similar to that for calculating the lift.

RECTANGULAR WINGS

The downwash factors for rectangular wings can not be found by the method which has just been used on elliptic wings, as the distributions of lift for a constant ratio of the effective to the induced angle of attack are not known for the rectangular wing. Nor do these factors depend on the aspect ratio of the rectangular wing only, but they will also depend on the distribution of the lift along the span. To calculate these factors for rectangular wings we assume several distributions of lift, calculate the entire lift and rolling moment, the downwash, and the corresponding geometric angle of attack; using this geometric angle of attack, we then calculate the ideal or fictitious lift and rolling moment. The downwash factors are then found by taking the ratio of the true lift and rolling moment to the computed fictitious lift and rolling moment.

The following calculations were made in order to find out how closely the downwash factors for rectangular wings agree with those already found for elliptic wings. The assumed distributions of lift for which the calculations have been made are given in Fig. 2.

(a) $L'=A\sqrt{1-\frac{x^2}{l^2}}$

(b) $L'=A\frac{x}{l}\sqrt{1-\frac{x^2}{l^2}}$

(c) $L'=A\frac{x^2}{l^2}\sqrt{1-\frac{x^2}{l^2}}$

(d) $L'=A\frac{x^3}{l^3}\sqrt{1-\frac{x^2}{l^2}}$

(e) $L'=A\left(1+\frac{x}{l}\right)\sqrt{1-\frac{x^2}{l^2}}=(a+b)$

(f) $L'=A\left(1+\frac{x^2}{l^2}\right)\sqrt{1-\frac{x^2}{l^2}}=(a+c)$

(g) $L'=A\left(\frac{x}{l}+\frac{x^2}{l^2}\right)\sqrt{1-\frac{x^2}{l^2}}=(b+c)$

(h) $L'=A\left(\frac{x}{l}+\frac{x^3}{l^3}\right)\sqrt{1-\frac{x^2}{l^2}}=(b+d)$

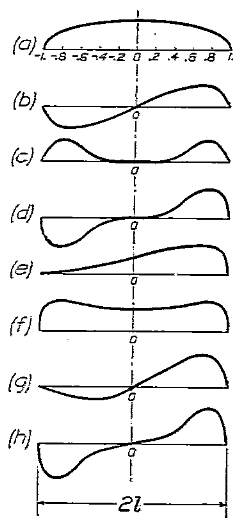

FIG. 2

where A is an arbitrary constant.

Since $L=\int_{-l}^{+l}L'\,dx$, and since $\int_{-l}^{+l}L'\,dx$ is represented by the algebraic sum of the areas between $L'=f(x)$ and the x-axis from $x=-l$ to $x=l$, the total lift in cases (b), (d), and (h) is zero. Likewise, since $M=\int_{-l}^{+l}xL'\,dx$, the total rolling moment is zero in cases (a), (c), and (f).

An outline of the calculation of the downwash factors for case (e) is given below. The calculations for the other cases are made in exactly the same way and will therefore not be given.

Calculation for case (e):

$$L'=A\left(1+\frac{x}{l}\right)\sqrt{1-\frac{x^2}{l^2}}$$

$$\frac{dL'}{dx}=\frac{A}{l^2}\left(\frac{l-x-\frac{2x^2}{l}}{\sqrt{1-\frac{x^3}{l^2}}}\right)$$

Substituting this value of $\frac{dL'}{dx}$ in the downwash formula

$$w=\frac{1}{4\pi\rho V}\int_{-l}^{+l}\frac{dL'}{dx}\frac{dx}{x'-x},$$

we have

$$w=\frac{A}{4\pi\rho Vl^2}\left(l\int_{-l}^{+l}\frac{dx}{(x'-x)\sqrt{1-\frac{x^2}{l^2}}}-\int_{-l}\frac{x\,dx}{(x'-x)\sqrt{1-\frac{x^2}{l^2}}}-\frac{2}{l}\int_{-l}^{+l}\frac{x^2\,dx}{(x'-x)\sqrt{1-\frac{x^2}{l^2}}}\right)$$

Putting $x=lu$, $dx=ldu$, we get

$$w=\frac{A}{4\pi\rho Vl^2}\left(l\int_{-1}^{+1}\frac{du}{\left(\frac{x'}{l}-u\right)\sqrt{1-u^2}}-l\int_{-1}^{+1}\frac{u\,du}{\left(\frac{x'}{l}-u\right)\sqrt{1-u^2}}-2l\int_{-1}^{+1}\frac{u^2\,du}{\left(\frac{x'}{l}-u\right)\sqrt{1-u^2}}\right).$$

The chief or principal values of these improper integrals are found to be:

$$\int_{-1}^{+1}\frac{du}{\left(\frac{x'}{l}-u\right)\sqrt{1-u^2}}=0,\ \int_{-1}^{+1}\frac{u\,du}{\left(\frac{x'}{l}-u\right)\sqrt{1-u^2}}=-\pi,\ \int_{-}^{+1}\frac{u^2\,du}{\left(\frac{x'}{l}-u\right)\sqrt{1-u^2}}=-\frac{\pi x'}{l}.$$

Hence the value of w is

$$w=\frac{A}{4\rho Vl^2}(l+2x),$$

where we have now replaced x' by x.

Let c_0 = chord of the rectangular wing.

Then

$$L'=2\pi\alpha_e qc_0=2\pi qc_0(\alpha_g-\alpha_i)$$

$$=2\pi qc_0\left(\alpha_g-\frac{w}{V}\right)=2\pi qc_0\left(\alpha_g-\frac{A}{4\rho V^2l^2}(l+2x)\right)$$

$$=2\pi qc_0\left(\alpha_g-\frac{A}{8ql^2}(l+2x)\right).$$

But

$$L'=A\left(1+\frac{x}{l}\right)\sqrt{1-\frac{x^2}{l^2}}.$$

Hence

$$A\left(1+\frac{x}{l}\right)\sqrt{1-\frac{x^2}{l^2}}=2\pi qc_0\left(\alpha_g-\frac{A}{8ql^2}(l+2x)\right),$$

from which we get

$$\alpha_g=A\left(\frac{\left(1+\frac{x}{l}\right)\sqrt{1-\frac{x^2}{l^2}}+\frac{\pi c_0}{4l^2}(l+2x)}{2\pi qc_0}\right).$$

The fictitious lift corresponding to this geometric angle of attack is

$$L_g=\int_{-l}^{+l}2\pi\alpha_g qc_0dx^{11}=2\pi qc_0A\int_{-}^{+}\left(\frac{\left(1+\frac{x}{l}\right)\sqrt{1-\frac{x^2}{l^2}}+\frac{\pi c_0}{4l^2}(l+2x)}{2\pi qc_0}\right)dx$$

$$=A\left(\int_{-l}\sqrt{1-\frac{x^2}{l^2}}dx+\frac{1}{l}\int_{-l}^{+l}\sqrt{1-\frac{x^2}{l^2}}\,xdx+\frac{\pi c_0}{4l^2}\int_{-l}^{+l}(l+2x)\,dx,\right.$$

or

$$L_g=\frac{A\pi l}{2}\left(1+\frac{c_0}{l}\right).$$

[11] From formula (6).

The fictitious rolling moment is

$$M_g=\int_{-l}^{+l} x\,L'_g dx = A\left(\int_{-l}^{+l}\sqrt{1-\frac{x^2}{l^2}}\,x dx+\frac{1}{l}\int_{-l}^{+l} x^2\sqrt{1-\frac{x^2}{l^2}}\,dx+\frac{\pi c_0}{4l^2}\int_{-l}^{+l}(lx+2x^2)\,dx\right),$$

or

$$M_g=\frac{A\pi l^2}{8}\left(1+\frac{8c_0}{3l}\right).$$

For the true lift we have

$$L=\int_{-l}^{+l} L'dx=\int_{-l}^{+l} A\left(1+\frac{x}{l}\right)\sqrt{1-\frac{x^2}{l^2}}\,dx=\frac{A\pi l}{2}:$$

and the true rolling moment is

$$M=\int_{-l}^{+l} xL'dx=A\int_{-l}^{+l}\left(x+\frac{x^2}{l}\right)\sqrt{1-\frac{x^2}{l^2}}\,dx=\frac{A\pi l^2}{8}.$$

Dividing L by L_g, we get

$$\frac{L}{L_g}=\frac{1}{1+\frac{c_0}{l}}=\frac{1}{1+\frac{2}{a}},\quad \text{since } a=\frac{2l}{c_0},$$

or

$$L=\left(\frac{1}{1+\frac{2}{a}}\right)L_g.$$

The downwash factor for lift in this case is therefore

$$f=\frac{1}{1+\frac{2}{a}}$$

Dividing M by M_g, we get

$$\frac{M}{M_g}=\frac{1}{1+\frac{8c_0}{3l}}=\frac{1}{1+\frac{16}{3a}}$$

The downwash factor for rolling moment is therefore

$$f=\frac{1}{1+\frac{16}{3a}}$$

The results of the calculations for the other assumed distributions of lift are given in Table I. In Table II are given the numerical values of the downwash factors for various aspect ratios, while Table III gives a comparison of rectangular and elliptic wings, the efficiency of elliptic wings being taken as 100.

The geometric angles of attack for both rectangular and elliptic wings for each of the lift distributions considered are given on the following page, and the graphs of these angles of attack are shown in Figure 3.

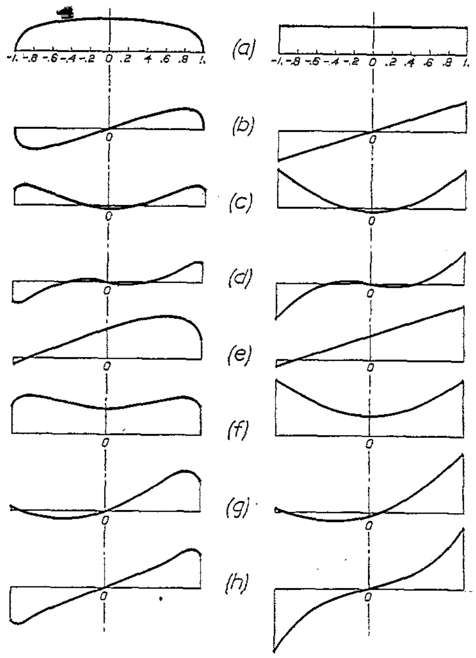

FIG. 3.—Graphs of geometric angles of attack for the various distributions of lift considered in section 4

GEOMETRIC ANGLES OF ATTACK

	RECTANGULAR WING	ELLIPTIC WING
(a)	$\alpha_g=\frac{A}{2\pi q c_0}\left(\sqrt{1-\frac{x^2}{l^2}}+\frac{\pi}{4}\frac{c_0}{l}\right)$	$\alpha_g=\frac{A}{2\pi q c_0}\left(1+\frac{\pi}{4}\frac{c_0}{l}\right)$
(b)	$\alpha_g=\frac{A}{2\pi q c_0}\frac{x}{l}\left(\sqrt{1-\frac{x^2}{l^2}}+\frac{\pi}{2}\frac{c_0}{l}\right)$	$\alpha_g=\frac{A}{2\pi q c_0}\frac{x}{l}\left(1+\frac{\pi}{2}\frac{c_0}{l}\right)$
(c)	$\alpha_g=\frac{A}{2\pi q c_0}\frac{x^2}{l^2}\left(\sqrt{1-\frac{x^2}{l^2}}+\frac{\pi}{4}\frac{c_0}{l}\left(3-\frac{1}{2}\frac{l^2}{x^2}\right)\right)$	$\alpha_g=\frac{A}{2\pi q c_0}\frac{x^2}{l^2}\left(1-\frac{\pi}{8}\frac{c_0 l}{x^2}+\frac{3\pi}{4}\frac{c_0}{l}\right)$
(d)	$\alpha_g=\frac{A}{2\pi q c_0}\frac{x^3}{l^3}\left(\sqrt{1-\frac{x^2}{l^2}}+\pi\frac{c_0}{l}\left(1-\frac{l^2}{4x^2}\right)\right)$	$\alpha_g=\frac{A}{2\pi q c_0}\frac{x^3}{l^3}\left(1+\pi\frac{c_0}{l}-\frac{\pi}{4}\frac{l c_0}{x^2}\right)$
(e)	$\alpha_g=\frac{A}{2\pi q c_0}\left(\left(1+\frac{x}{l}\right)\sqrt{1-\frac{x^2}{l^2}}+\frac{\pi}{4}\frac{c_0}{l}\left(1+\frac{2x}{l}\right)\right)$	$\alpha_g=\frac{A}{2\pi q c_0}\left(\left(1+\frac{x}{l}\right)+\frac{\pi}{4}\frac{c_0}{l}\left(1+\frac{2x}{l}\right)\right)$
(f)	$\alpha_g=\frac{A}{2\pi q c_0}\left(\left(1+\frac{x^2}{l^2}\right)\sqrt{1-\frac{x^2}{l^2}}+\frac{\pi}{4}\frac{c_0}{l}\left(\frac{1}{2}+\frac{3x^2}{l^2}\right)\right)$	$\alpha_g=\frac{A}{2\pi q c_0}\left(\left(1+\frac{x^2}{l^2}\right)+\frac{\pi}{4}\frac{c_0}{l}\left(\frac{1}{2}+\frac{3x^2}{l^2}\right)\right)$
(g)	$\alpha_g=\frac{A}{2\pi q c_0}\left(\left(\frac{x}{l}+\frac{x^2}{l^2}\right)\sqrt{1-\frac{x^2}{l^2}}+\frac{\pi}{4}\frac{c_0}{l}\left(-\frac{1}{2}+\frac{2x}{l}+\frac{3x^2}{l^2}\right)\right)$	$\alpha_g=\frac{A}{2\pi q c_0}\left(\left(\frac{x}{l}+\frac{x^2}{l^2}\right)+\frac{\pi}{4}\frac{c_0}{l}\left(-\frac{1}{2}+\frac{2x}{l}+\frac{3x^2}{l^2}\right)\right)$
(h)	$\alpha_g=\frac{A}{2\pi q c_0}\left(\left(\frac{x}{l}+\frac{x^3}{l^3}\right)\sqrt{1-\frac{x^2}{l^2}}+\frac{\pi}{4}\frac{c_0}{l}\left(\frac{x}{l}+\frac{4x^3}{l^3}\right)\right)$	$\alpha_g=\frac{A}{2\pi q c_0}\left(\left(\frac{x}{l}+\frac{x^3}{l^3}\right)+\frac{\pi}{4}\frac{c_0 x}{l^2}\left(1+\frac{4x^2}{l^2}\right)\right)$

These columns give the distribution of geometric angle of attack to show the lift distribution specified It will be observed that the geometric angles of attack for a given distribution of lift are identical for the two wing forms, except that the radical term $\sqrt{1-\frac{x^2}{l^2}}$ in the case of rectangular wings is replaced by 1 in the case of elliptic wings; and since this radical is unity only at the middle of the wing span, it is plain that the geometric angle of attack necessary for a given lift must be greater in the case of elliptic wings than in the case of rectangular wings.

TABLE I

Lift distribution	Downwash factors for lift		Downwash factors for rolling moment	
	Rectangular wing	Elliptic wing	Rectangular wing	Elliptic wing
$L'=A\sqrt{1-\frac{x^2}{l^2}}$	$f_1=\frac{1}{1+\frac{2}{a}}$		(1)	
$L'=A\frac{x}{l}\sqrt{1-\frac{x^2}{l^2}}$	(2)		$f_3=\frac{1}{1+\frac{16}{3a}}$	
$L'=A\frac{x^2}{l^2}\sqrt{1-\frac{x^2}{l^2}}$	$f_2=\frac{1}{1+\frac{4}{a}}$		(1)	
$L'=A\frac{x^3}{l^3}\sqrt{1-\frac{x^2}{l^2}}$	(2)		$f_4=\frac{1}{1+\frac{112}{15a}}$	
$L'=A\left(1+\frac{x}{l}\right)\sqrt{1-\frac{x^2}{l^2}}$	$f_1=\frac{1}{1+\frac{2}{a}}$	$f_1=\frac{1}{1+\frac{2}{a}}$ (all distributions)	$f_3=\frac{1}{1+\frac{16}{3a}}$	$f_2=\frac{1}{1+\frac{4}{a}}$ (all distributions)
$L'=A\left(1+\frac{x^2}{l^2}\right)\sqrt{1-\frac{x^2}{l^2}}$	$f_5=\frac{1}{1+\frac{12}{5a}}$		(1)	
$L'=A\left(\frac{x}{l}+\frac{x^2}{l^2}\right)\sqrt{1-\frac{x^2}{l^2}}$	$f_2=\frac{1}{1+\frac{4}{a}}$		$f_3=\frac{1}{1+\frac{16}{3a}}$	
$L'=A\left(\frac{x}{l}+\frac{x^3}{l^3}\right)\sqrt{1-\frac{x^2}{l^2}}$	(2)		$f_6=\frac{1}{1+\frac{272}{45a}}$	

[1] Rolling moment is zero for this distribution of lift.
[2] Lift is zero for this distribution of lift.

In the above table $a=$ aspect ratio $=\frac{\text{area}}{\text{span}^2}=\frac{2l}{c_0}$ for rectangular wings, $\frac{8l}{\pi c_0}$ for elliptic wings.

TABLE II

NUMERICAL VALUES OF THE DOWNWASH FACTORS FOR VARIOUS ASPECT RATIOS

Lift	$f \diagdown a$	4	5	6	7	8	9	10	L'
	f_1	0.667	0.714	0.750	0.778	0.800	0.818	0.833	Elliptic wing, all distributions. $A\sqrt{1-\frac{x^2}{l^2}}$ for rectang.
	f_5	.625	.676	.714	.745	.770	.790	.807	$A\left(1+\frac{x^2}{l^2}\right)\sqrt{1-\frac{x^2}{l^2}}$ for rectang.
	f_2	.500	.556	.600	.636	.667	.692	.714	$A\left(\frac{x}{l}+\frac{x^2}{l^2}\right)\sqrt{1-\frac{x^2}{l^2}}$ for rectang.
Moment	$a'=$	$5\frac{1}{3}$	$6\frac{2}{3}$	8	$9\frac{1}{3}$	$10\frac{2}{3}$	12	$13\frac{1}{3}$	
	f_2	.500	.556	.600	.636	.667	.692	.714	Elliptic wing, all distributions.
	f'_3	.500	.556	.600	.636	.667	.692	.714	$A\frac{x}{l}\sqrt{1-\frac{x^2}{l^2}}$ for rectang.
	f'_6	.469	.524	.569	.607	.638	.665	.688	$A\left(\frac{x}{l}+\frac{x^3}{l^3}\right)\sqrt{1-\frac{x^2}{l^2}}$ for rectang.
	f'_4	.417	.472	.517	.556	.588	.617	.641	$A\frac{x^3}{l^3}\sqrt{1-\frac{x^2}{l^2}}$ for rectang.

$a=$	$\frac{\pi}{2}$	π	2π	4.5π	Rectangular cylindrical wing.[12]
$f_b=$	0.427	0.588	0.730	.847	
$f_b/f_1=$	.970	.965	.962	.966	

a denotes the aspect ratio, b^2/s.
a' denotes the aspect ratio of an elliptic wing having the same value I/b^4 as the wing in question.
f refers to a and f' to a', which means that either a or a' has to be inserted in the formula for f or f'.

[12] A. Betz, Beiträge zur Tragflügeltheorie mit besouderer Berücksichtigung des einfachen Rechteckigen Flügels. Dissertation, Göttingen, 1919.

TABLE III

CORRECTION OF THE DOWNWASH FACTORS FOR LIFT, REFERRING TO THE ASPECT RATIO a OF AN ELLIPTIC WING OF EQUAL SPAN AND AREA

Lift	a=	4	5	6	7	8	9	10	Case
	f_1/f_1	1.000	1.000	1.000	1.000	1.000	1.000	1.000	(a)
	f_5/f_1	.937	.947	.952	.958	.963	.966	.968	(f)
	f_2/f_1	.750	.779	.800	.818	.834	.846	.857	(c)

TABLE IV

CORRECTION OF THE DOWNWASH FACTORS FOR ROLLING MOMENT, REFERRING TO THE ASPECT RATIO a' OF AN ELLIPTIC WING OF EQUAL SPAN AND MOMENT OF INERTIA

Moment	a of rectangular wing=	4	5	6	7	8	9	10	Case
	a' of elliptic wing=	5⅓	6⅔	8	9⅓	10⅔	12	13⅓	
	f'_3/f_2	1.000	1.000	1.000	1.000	1.000	1.000	1.000	(b)
	f'_6/f_2	.938	.942	.948	.954	.957	.961	.964	(h)
	f'_4/f_2	.834	.848	.863	.874	.882	.892	.898	(d)

CONCLUSION

The shape of actual airplane wings is somewhere between an ellipse and a rec tangle Hence the error committed by applying to them the formulas valid for the ellipse will be about half as large on the average as computed in this paper.

The aspect ratio of monoplanes is in the neighborhood of 6. The distribution of lift is then approximately like (a), and then the downwash factor for the lift would be exact. The results of Dr. Betz for rectangular wings without warp indicate an error of about 2 per cent. A distribution much more different from that of an elliptical wing than the actual distribution is the distribution (f). Even this distribution for ordinary wings has a downwash factor only 2½ per cent different from the universal factor derived from the elliptical shape.

The downwash factor for the rolling moment is of interest for the computation of the aileron moments and for the investigation of the air forces in certain maneuvers, chiefly during a roll, which can be approximately obtained by substituting an equivalent warp of the wings.

The latter is best represented by the distribution (b). Then there is no error in substituting the downwash factor of the elliptical wing of equal span and moment of inertia of the wing surface with respect to the axis.

The displacement of the ailerons gives a distribution of lift which might be well represented by (d). The error for actual wings would then appear to be approximately 7 per cent. By so much the actual rolling moment is smaller than the rolling moment computed by using the downwash factor of the ellipse of equal "inertia ratio" I/b^4.